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Letter to the Editor

Free vibration of hollow bodies of revolution

D. Redekop*

Department of Mechanical Engineering, University of Ottawa, Ottawa, Canada K1N 6N5

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1. Introduction

Hollow bodies of revolution are of considerable engineering importance, finding applications in pressure vessels, piping, machinery, etc. To find accurate values for the natural frequencies of vibration for such bodies recourse must be made to the three-dimensional theory of elasticity. Solutions for hollow circular cylinders have been obtained in a number of studies [1–4], but few results are available for other hollow bodies of revolution.

In the present work the three-dimensional theory of elasticity is used to set up an accurate solution for the natural frequencies of vibration of a hollow body of revolution of arbitrary geometry. A semi-analytical approach is adopted, in which solutions are obtained for specified circumferential harmonic modes of vibration. The new differential quadrature method (DQM) [5,6], which is versatile with regards to boundary conditions, is used to obtain accurate numerical results. Validation is through comparison with previously published results for simply supported and fixed hollow cylinders. Finally results are presented for a hollow hemisphere, and conclusions are drawn.

2. Theory

For a linear isotropic three-dimensional solid the equations of equilibrium are given [7] as

$$\begin{aligned} L_{11}u_1 + L_{12}u_2 + L_{13}u_3 + K_1/G &= 0,\\ L_{21}u_1 + L_{22}u_2 + L_{23}u_3 + K_2/G &= 0,\\ L_{31}u_1 + L_{32}u_2 + L_{33}u_3 + K_3/G &= 0, \end{aligned} \tag{1}$$

where the differential operators L_{ij} are functions of v, E, H_i. The material properties v, E, respectively, are the Poisson ratio and Young's modulus, and the H_i are the Lamé coefficients given by $H_i = \sqrt{(\mathbf{R}_{,i}.\mathbf{R}_{,i})}$. Here $\mathbf{R}$ is the position vector, and a comma subscript indicates differentiation with respect to the variable that follows. The K_i are the body forces per unit

*Corresponding author. Tel.: +1-613-5625800x6290; fax: +1-613-652-5177.
E-mail address: dredekop@tesla.cc.uottawa.ca (D. Redekop).

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volume, and $G = E/[2(1+v)]$ is the shear modulus. The position variables q_i are chosen such that $q_1 = \alpha$, $q_2 = \beta$ locate a point in the radial plane, while $q_3 = \theta$ defines the angular position of that plane about the axis of symmetry (Fig. 1). With the assumption of axisymmetry of geometry the H_i are functions of the variables α, β only. The displacement components u_1, u_2, and u_3 are, respectively, in the α, β, and θ directions.

With application of the D'Alembert principle the body forces $K_i = -\rho \ddot{u}_i$ introduce 2nd order time derivatives of the displacements and the mass density ρ into Eq. (1). Assuming cyclical vibrations the displacement components are taken as

$$u_1 = U_1(\alpha, \beta)\cos n\theta \cos \omega t,$$
$$u_2 = U_2(\alpha, \beta)\cos n\theta \cos \omega t,$$
$$u_3 = U_3(\alpha, \beta)\sin n\theta \cos \omega t, \tag{2}$$

where n is the number of the circumferential harmonic, ω the natural frequency, and t the time. Substitution of the expressions (2) into Eq. (1) leads to three homogeneous differential equations for the three displacement functions $U_i, i = 1, 2, 3$, and the frequency ω. Mathematically a two-dimensional eigenvalue problem is defined, for each choice of n, in the variables α, β.

A solution is obtained for a hollow body whose radial surfaces are stress-free. For these surfaces, defined by $\alpha = cnst$, the relations $\sigma_1 = \sigma_{12} = \sigma_{13} = 0$ are satisfied. Conditions on the ends of the body are considered herein as either simply supported or fixed. For the simply supported end conditions, on an end defined by $\beta = cnst$, the relations $u_2 = \sigma_{21} = \sigma_{23} = 0$ are satisfied. For the fixed end conditions the relations $u_1 = u_2 = u_3 = 0$ are satisfied.

The boundary value problem is solved using the DQM for a typical harmonic n. A two-dimensional mesh of sampling points is defined in the radial (α, β) plane, with spacing based on the Chebyshev–Gauss–Lobatto system [5,6]. Domain relations are satisfied at interior sampling points, and boundary relations at boundary sampling points. Derivatives of the displacement

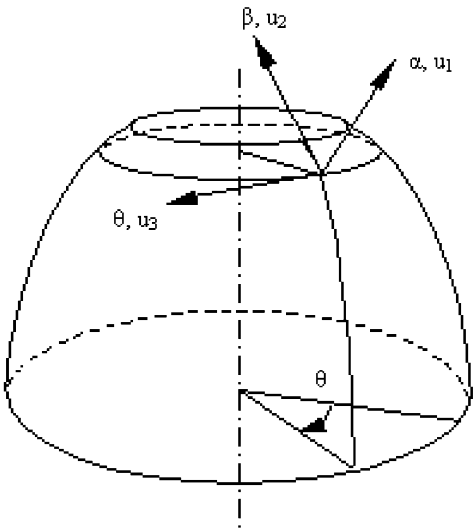

Fig. 1. Co-ordinates and displacements.

functions in a given direction are replaced by the weighted sums of the values of the function at the sampling points of the mesh in a line following the given direction [5,6]. The weighting functions used herein correspond to the selection of a power series for the trial functions [5,6].

Application of the quadrature rules of the DQM allows the problem of differential equations to be transformed into one of simultaneous linear equations. This set of equations leads to a matrix equation of the form $[K](u) = \lambda[M](u)$, where the unknowns (u) are the values of the displacements at the sampling points, λ is the eigenvalue dependent on ω, and $[K]$, $[M]$ are known matrices. The equation is solved using a general eigenvalue routine.

The derivation of the governing equation described herein, although lengthy, allows for the analysis of hollow bodies of revolution of arbitrary shape. The appropriate values of the Lamé coefficients H_i are inserted only at the solution stage in enforcing either the domain or boundary conditions at the sampling points. Thus the equations derived can readily be used for hollow bodies of cylindrical, spherical, toroidal, etc. shape. Furthermore the DQM solution is not tied to a specific set of boundary conditions, as is the case for most 'series solutions'.

3. Validation and results

In this section two validation examples are presented which involve hollow circular cylinders, and additionally results are given for a hollow hemisphere. For all solutions given herein a DQM mesh of 19×19 sampling points was used, and the material properties were taken as $v = 0.3$, $E = 0.2e12$ Pa, and $\rho = 7800$ kg/m^2. Results given are for a frequency parameter Ω defined as $\Omega = K\omega$, where K is a constant defined in the following.

For the first validation example results are obtained for four geometric cases M, having geometry as follows: $M = 1$, $h/R = 0.2$, $h/L = 0.3$; $M = 2$, $h/R = 0.2$, $h/L = 0.7$; $M = 3$, $h/R = 0.5$, $h/L = 0.3$; $M = 4$, $h/R = 0.5$, $h/L = 0.7$. Here h, R, L are, respectively, the thickness, mean radius, and length of the hollow cylinder. Boundary conditions on the ends are of the simply supported kind. Results from a Fourier–Bessel 'series solution' have been given previously for these hollow circular cylinders by Armenakas et al. [1]. Their results however did not include any plane strain modes.

Table 1 gives a comparison of results obtained using the present method with results given in Ref. [1]. For each of the four geometric cases M the frequency parameter Ω is given for the first six modes for each of the first two circumferential harmonics n. Following Ref. [1] K was taken as $K = h/(\pi v_2)$, where $v_2 = \sqrt{G/\rho}$. It is seen that the present DQM approach gives, for each of the natural frequencies cited in Ref. [1], results having differences less than 0.1%.

For the second validation example results are obtained for two geometric cases M, having geometry as follows; $M = 1$, $L/R_1 = 3.0$, $R_0/R_1 = 0.5$; $M = 2$, $L/R_1 = 6.0$, $R_0/R_1 = 0.5$. Here L, R_0, R_1 are, respectively, the length, inside radius, and outside radius of the cylinder. Results identified by 1S, 2S give symmetric modes in the axial direction while results identified by 1A, 2A give antisymmetric modes. Boundary conditions on the ends are of the fixed kind. A 'series solution' for these hollow cylinders was determined previously by Zhou et al. [4] using a Chebyshev–Ritz approach.

Table 2 gives a comparison of results from Zhou et al. [4] with results obtained using the DQM. For each of the two geometric cases M the frequency parameter Ω is given for the first six

Table 1
Comparison of results for Ω with Fourier–Bessel (FB) method [1]

M	n	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1	1	FB	0.15516	0.30799	0.51578	1.05309	1.14534	1.77791
	1	DQM	0.15514	0.30798	0.51578	1.05308	1.14533	1.77791
	2	FB	0.15826	0.32960	0.54745	1.05851	1.16291	1.77222
	2	DQM	0.15827	0.32960	0.54745	1.05851	1.16290	1.77221
2	1	FB	0.48606	0.70302	1.11963	1.22879	1.55996	1.76830
	1	DQM	0.48606	0.70302	1.11962	1.22879	1.55999	1.76830
	2	FB	0.49337	0.71199	1.13000	1.23449	1.57037	1.77149
	2	DQM	0.49337	0.71199	1.12999	1.23448	1.57037	1.77148
3	1	FB	0.20903	0.37185	0.57911	1.09336	1.20288	1.78139
	1	DQM	0.20902	0.37184	0.57909	1.09343	1.20284	1.78257
	2	FB	0.21091	0.46672	0.72316	1.12457	1.32894	1.75785
	2	DQM	0.21091	0.46672	0.72315	1.12456	1.32894	1.75784
4	1	FB	0.52886	0.72299	1.13966	1.27719	1.59566	1.78233
	1	DQM	0.52886	0.72299	1.13966	1.27720	1.59564	1.78228
	2	FB	0.55149	0.78202	1.18072	1.32497	1.66703	1.80911
	2	DQM	0.55149	0.78202	1.18072	1.32496	1.66702	1.80910

symmetric and antisymmetric modes for each of the first two circumferential harmonics n. The factor K was taken as $K = 2h/v_2$ where $h = R_1 - R_0$. It is seen that the present method gives, for each of natural frequencies cited in Ref. [4], results having agreement to four or five figures. A comparison of truncated values from the present study rather than rounded values would lead to even better agreement.

A final set of results is determined for the case of a hollow hemisphere. Results obtained are compared with results determined using the finite element method (FEM). The results are found for two geometric cases M, having geometry as follows; $M = 1$, $h/R = 0.2$; $M = 2$, $h/R = 0.5$. Here h and R are, respectively, the thickness and mean radius of the hollow hemisphere. Results identified by 1S, 2S are for simply supported end conditions (at $\beta \equiv \phi = 0$, ϕ_u), while results identified by 1F, 2F are for fixed end conditions. To avoid the singularity at $\phi = \pi/2$, an upper boundary is assumed in the radial plane at $\phi_u = \pi/2 - \delta$, where $\delta = 10^{-3}$.

Table 3 gives a comparison of results from the present DQM method with results obtained using the FEM. For each of the two boundary conditions of the geometric cases M the frequency parameter Ω is given for the first six modes for each of the first two circumferential harmonics n. The factor K was taken as $K = h/(\pi v_2)$. It is seen that the results from the two methods differ substantially for the first mode, especially for the $M = 1, 2$ $(n = 1)$ cases. Otherwise agreement to within 2% is obtained, and agreement is generally excellent for the $n = 2$ results. Frequencies increase as the thickness is increased, and as the boundary conditions are changed from simply-supported to fixed. For the cases considered the frequencies for the harmonic $n = 1$ are lower than those for the harmonic $n = 2$.

Table 2
Comparison of results for Ω with Chebyshev–Ritz (CR) method [4]

M	n	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1S	1	CR	1.6109	2.1297	3.3630	3.4760	4.4826	5.3087
	1	DQM	1.6109	2.1297	3.3630	3.4760	4.4826	5.3087
	2	CR	1.8288	3.0645	3.5645	4.4679	5.2467	5.4840
	2	DQM	1.8288	3.0645	3.5645	4.4679	5.2467	5.4839
2S	1	CR	0.7054	1.5343	1.6318	2.4485	2.5355	3.0966
	1	DQM	0.7054	1.5343	1.6319	2.4486	2.5356	3.0966
	2	CR	1.1747	1.7648	2.5797	2.7846	3.3500	3.5180
	2	DQM	1.1747	1.7649	2.5798	2.7846	3.3501	3.5185
1A	1	CR	0.8155	2.5304	2.8947	3.4640	4.3033	4.3915
	1	DQM	0.8155	2.5304	2.8947	3.4640	4.3033	4.3915
	2	CR	1.2396	2.6409	3.7995	4.5122	4.5569	5.3062
	2	DQM	1.2396	2.6409	3.7995	4.5122	4.5569	5.3062
2A	1	CR	0.3395	1.1338	1.9880	2.0930	2.8775	2.9025
	1	DQM	0.3395	1.1338	1.9880	2.0931	2.8776	2.9026
	2	CR	1.0288	1.4286	2.1562	2.9908	3.0797	3.7107
	2	DQM	1.0288	1.4286	2.1563	2.9908	3.0797	3.7107

Table 3
Results for Ω for hollow hemisphere

M	n	Theory	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1S	1	FEM	0.0337	0.1069	0.1411	0.1769	0.2122	0.2740
	1	DQM	0.0247	0.1057	0.1378	0.1747	0.2113	0.2716
	2	FEM	0.0780	0.1336	0.1996	0.2193	0.2827	0.3273
	2	DQM	0.0780	0.1335	0.1996	0.2189	0.2876	0.3266
2S	1	FEM	0.0873	0.3286	0.3656	0.5004	0.6570	0.6830
	1	DQM	0.0625	0.3244	0.3553	0.4959	0.6405	0.6532
	2	FEM	0.2181	0.4778	0.4936	0.6490	0.7951	0.8129
	2	DQM	0.2181	0.4757	0.4866	0.6466	0.7871	0.7927
1F	1	FEM	0.0876	0.1346	0.1918	0.2159	0.2503	0.3401
	1	DQM	0.0821	0.1334	0.1873	0.2143	0.2481	0.3416
	2	FEM	0.1138	0.1834	0.2306	0.2819	0.3121	0.3942
	2	DQM	0.1137	0.1829	0.2304	0.2824	0.3115	0.3970
2F	1	FEM	0.2422	0.4028	0.5146	0.5774	0.8025	0.8226
	1	DQM	0.2279	0.3992	0.5055	0.5753	0.7956	0.8130
	2	FEM	0.3627	0.5362	0.6539	0.7264	0.9220	0.9570
	2	DQM	0.3668	0.5360	0.6532	0.7259	0.9216	0.9549

4. Conclusions

A solution based on the differential quadrature method has been developed for the free vibration problem of arbitrary hollow bodies of revolution. The solution has an accuracy approaching that of 'series solutions', but has greater flexibility with respect to boundary conditions. With the semi-analytical approach that is used the numerical effort required is much less than that needed for a full three-dimensional finite element approach. Work is currently underway to extend the present approach to anisotropic hollow bodies of revolution.

References

[1] A.E. Armenakas, D.C. Gazis, G. Herrmann, *Free Vibrations of Circular Cylindrical Shells*, Pergamon, Oxford, 1969.
[2] R.K. Singal, K. Williams, A theoretical and experimental study of vibrations of thick circular cylindrical shells and rings, *Journal of Vibration, Stress, and Reliability in Design* 110 (1988) 533–537.
[3] J.Y. So, A.W. Leissa, Free vibrations of thick hollow circular cylinders from three-dimensional analysis, *Journal of Vibration and Acoustics* 119 (1997) 89–95.
[4] D. Zhou, Y.K. Cheung, S.H. Lo, F.T.K. Au, 3D vibration analysis of solid and hollow circular cylinders via Chebyshev-Ritz method, *Computer Methods in Applied Mechanics and Engineering* 192 (2003) 1575–1589.
[5] W. Jiang, D. Redekop, Polar axisymmetric vibration of a hollow toroid using the DQM, *Journal of Sound and Vibration* 251 (2002) 761–765.
[6] C. Shu, *Differential Quadrature and its Application in Engineering*, Springer, Berlin, 2000.
[7] A.I. Lur'e, *Three-Dimensional Problems of the Theory of Elasticity*, Wiley, New York, 1964.